Tentamen Functionaalanalyse
10/11/04

1. Let $F : L^2[0, \pi] \to \mathbb{C}$ be defined by

$$F(f) := \int_0^\pi \sin(t) f(t) dt, \qquad f \in L^2[0, \pi].$$

   (a) Is F linear? Justify the answer!
   (b) Show that F is bounded. Determine $\|F\|$.
   (c) Let $G : L^2[0, \pi] \to \mathbb{C}$ be a bounded linear functional defined on $L^2[0, \pi]$. Does there exist some $g \in L^2[0, \pi]$ such that G is of the form

$$G(f) = \pi^2 \int_0^\pi e^{it} f(t) g(t) dt, \quad f \in L^2[0, \pi]?$$

   Justify the answer!

2. Solve the integral equation

$$x(t) + \int_0^\pi x(s) \cos(t - s) ds = \cos t + 3 \sin t, \quad x \in C[0, \pi].$$

3. Let E be an infinite-dimensional normed space. Let $x, y \in E$ be linearly independent with $\|x\| = \|y\| = 1$, and let $U = \text{span}\,\{x, y\}$. For $\beta \in \mathbb{C}$, let $\ell_\beta : U \to \mathbb{C}$ be the linear functional on U such that $\ell_\beta(x) = 3i + 2$ and $\ell_\beta(y) = \beta$.

   Does there exist some $\beta_0 \in \mathbb{C}$ such that $|\ell_{\beta_0}(z)| \leq \sqrt{13}\,\|z\|$ for all $z \in U$?

   Does there exist some $L \in E'$ (E' is the dual space of E) and some $\beta \in \mathbb{C}$ such that ℓ_β is the restriction of L on U: $L|_U = \ell_\beta$, and

   (a) $\|L\| = 3$?
   (b) $\|L\| = \sqrt{13}$?
   (c) $\|L\| \geq 23$?

   Justify the answers!

4. Provide the linear space $C[0, 1]$ with

$$\|x\| := \|x\|_\infty + 2 \int_0^1 |x(t)| dt + 3|x(1)|, \quad x \in C[0, 1].$$

   (a) Show that $\|\cdot\|$ is a norm on $C[0, 1]$.
   (b) Show that $C[0, 1]$ with the norm $\|\cdot\|$ is a complete space.
   (c) Show that the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent on $C[0, 1]$.